Section 3.4

Constrained extrema & Lagrange multipliers

Suppose that you want to minimize the surface area of a can, subject to keeping the volume fixed (for example, to minimize the cost of the materials).

Or, suppose that a particle moves along a curve in a container where the force (or temp, or pressure) is given by some function and we want to find the max/min force that the particle experiences.

In other words, we want to solve

(maximize)
minimize $f(\vec{x})$
subject to $g(\vec{x}) = c$

e.g. minimize $2\pi r h + 2\pi r^2$

subject to $\pi r^2 h = 1$

Theorem (the method of Lagrange multipliers)

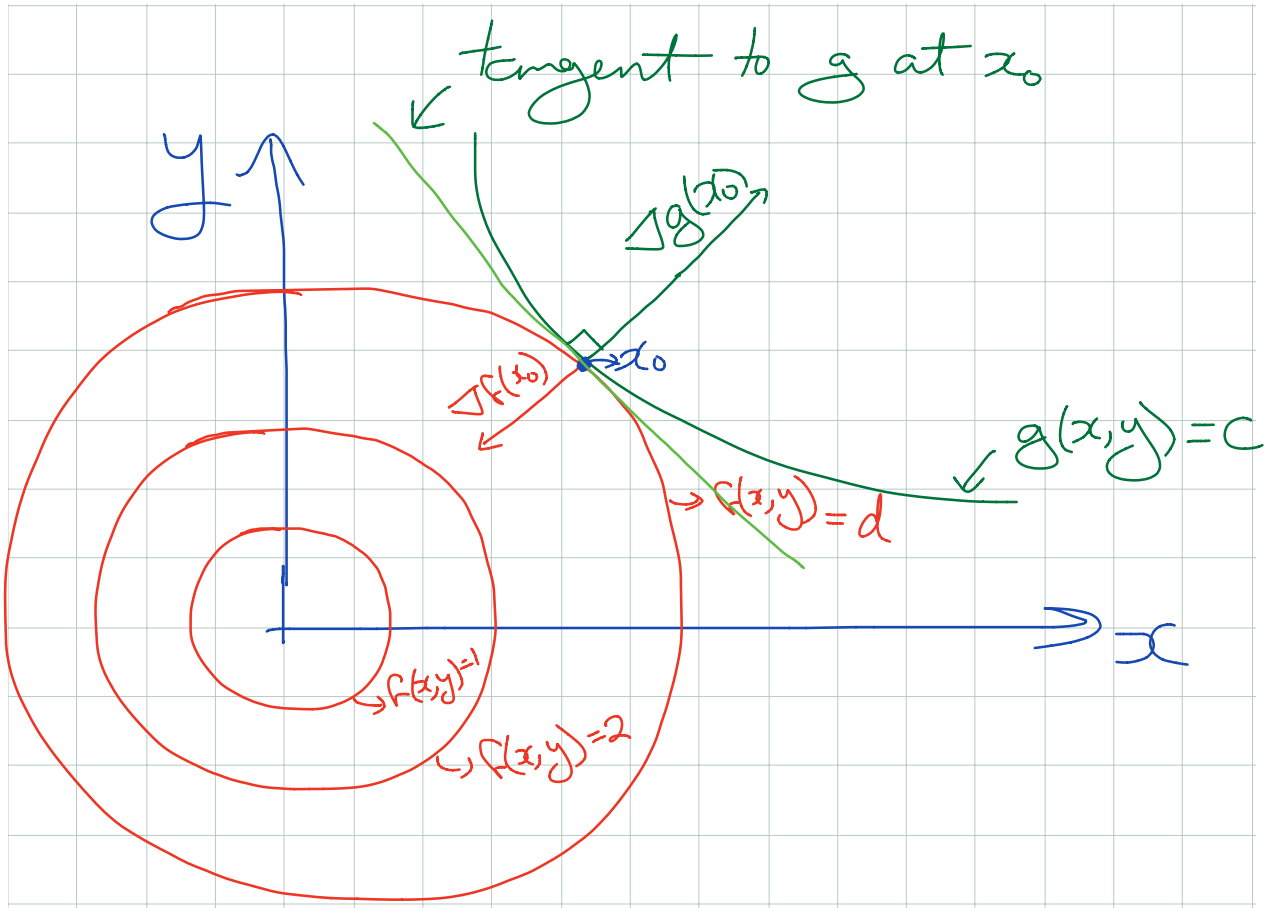
Let $f: U \subset \mathbb{R}^n \to \mathbb{R}$ & $g: U \subset \mathbb{R}^n \to \mathbb{R}$ be C^1 Functions.

- Let $\vec{x}_0 \in U$ & $g(\vec{x}_0) = c$, $\nabla g(\vec{x}_0) \neq 0$.
- Let S = level set for g with value c
$= \{ \vec{x} \in \mathbb{R}^n, g(\vec{x}) = c \}$

If f restricted to S (denoted $f|_S$) has a local max or min, then there is a number λ such that (possibly $\lambda = 0$)

$$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$$

We call x_0 a critical pt of $f|_S$.



But what does this mean for us?

Suppose that f is continuous and the constraint is closed and bounded.

From the first theorem, we have

$\nabla f(\vec{x}_0) = \lambda \nabla g(\vec{x}_0)$ at local max or min x_0

we just need to find the points (for some λ) that satisfy

$$\nabla f(\vec{x}) = \lambda \nabla g(\vec{x})$$

— lagrange equations (or lagrange condition)

and check to see if they are minima or maxima or neither

So: Step 1: Write the Lagrange Equations

$$\nabla f = \lambda \nabla g$$

Step 2: Solve for λ, x, y, z

Step 3: Compute f at the critical pts. and select the min & max.

Remark: If the constraint is not closed & bounded the min or max may not exist. eg. $f(x,y) = x^2+y^2$ s.t. $x+y=1$

Example: Find the extrema of $f(x,y) = x^2-y^2$ on the circle $x^2+y^2=1$

Soln. • $g(x,y) = x^2+y^2$

Step 1 $\nabla f(x,y) = (2x, -2y)$

$\nabla g(x,y) = (2x, 2y)$

Step 2

So we must find λ and x and y s.t.

$(2x, -2y) = \lambda(2x, 2y)$ & $x^2+y^2=1$

$\Rightarrow x = \lambda x \Rightarrow x = 0$ or $\lambda = 1$

If $x = 0$

$x^2 + y^2 = 1 \Rightarrow y = \pm 1$

$y = -\lambda y \Rightarrow \lambda = -1$

If $\lambda = 1$

$y = -\lambda y \Rightarrow y = 0$

& $x = \pm 1$

So we get the pts $(0,1), (0,-1), (1,0)$ & $(-1,0)$

Step 3

we can now check them to see if they are maxima or minima

$\Rightarrow$ max is $f(1,0) = f(-1,0) = 1$

min is $f(0,1) = f(0,-1) = -1$

—x—

Example

Find the minimum & maximum of $f(x,y,z) = x+y+z$ subject to $x^2 + 2y^2 + 3z^2 = 6$

Soln

Step 1

$\nabla f(x,y,z) = (1, 1, 1)$

$\nabla g(x,y,z) = (2x, 4y, 6z)$

Want $\nabla f = \lambda \nabla g$

So we want $\begin{cases} 1 = 2\lambda x \\ 1 = 4\lambda y \\ 1 = 6\lambda z \\ x^2 + 2y^2 + 3z^2 = 6 \end{cases}$

4 eq'ns → 4 unknowkn

Step 2 (Solve for x, y, z, λ)

So $\lambda x = 2\lambda y = 3\lambda z$

and $\lambda \neq 0$ (otherwise $1 = 2\lambda x$ is not satisfied

this $\boxed{x = 2y = 3z}$ (*)

$\Rightarrow x^2 = 4y^2 = 9z^2$

$\Rightarrow y^2 = x^2/4$ & $z^2 = x^2/9$

So $x^2 + \frac{2x^2}{4} + \frac{3x^2}{9} = 6 \Leftrightarrow (3/2 + 1/3)x^2 = 6$

$\Leftrightarrow \frac{11}{6}x^2 = 6 \Rightarrow x^2 = 36/11$

$x_1 = \frac{6}{\sqrt{11}}$, $x_2 = \frac{-6}{\sqrt{11}}$

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by (*) $y_1 = \frac{x_1}{2} = \frac{3}{\sqrt{11}}$ $y_2 = \frac{x_2}{2} = \frac{-3}{\sqrt{11}}$

$z_1 = \frac{x_1}{3} = \frac{2}{\sqrt{11}}$ $z_2 = \frac{x_2}{3} = \frac{-2}{\sqrt{11}}$

$P_1 = \frac{1}{\sqrt{11}}(6, 3, 2)$ & $P_2 = \frac{1}{\sqrt{11}}(-6, -3, -2)$

Step 3

Finally, checking P_1 & P_2, we see that $f(P_1) \geq f(P_2)$. Moreover, our constraint was closed & bounded

So $f(P_1)$ is max

$f(P_2)$ is min.



Chapter 4

Sec 4.1 Acceleration

Recall: Given a path $\vec{c}(t) = (x(t), y(t), z(t))$

we can compute $\vec{c}\,'(t) = (x'(t), y'(t), z'(t))$

$\vec{v}(t)$

↖ velocity vector

Recall: $\vec{c}\,'(t_0)$ is the tangent vector to the path at the point $\vec{c}(t_0)$.

Recall: $\|\vec{c}\,'(t)\|$ is the speed.

Differentiation rules: $\vec{b}: \mathbb{R} \to \mathbb{R}^3$, $\vec{c}: \mathbb{R} \to \mathbb{R}^3$

- $\dfrac{d}{dt}\left(\vec{b}(t) + \vec{c}(t)\right) = \vec{b}'(t) + \vec{c}'(t)$.
- If $f(t): \mathbb{R} \to \mathbb{R}$

$$\left(f(t)\,\vec{c}(t)\right)' = f'(t)\,\vec{c}(t) + f(t)\,\vec{c}'(t)$$

e.g. $f(t) = t$, $\vec{c}(t) = (t, t^2, t^3)$ $\Rightarrow \left(f(t)\,\vec{c}(t)\right)' = 1\cdot(t, t^2, t^3) + t(1, 2t, 3t^2)$

$$= (2t, 3t^2, 4t^3)$$

- $\left(\vec{b}(t)\cdot\vec{c}(t)\right)' = \vec{b}'(t)\cdot\vec{c}(t) + \vec{b}(t)\cdot\vec{c}'(t)$

exercise: prove it

- $\left(\vec{b}(t)\times\vec{c}(t)\right)' = \vec{b}'(t)\times\vec{c}(t) + \vec{b}(t)\times\vec{c}'(t)$
- $\left(\vec{c}(f(t))\right)' = \vec{c}'(f(t))\,f'(t)$

Example: If $\|\vec{c}(t)\| = \text{const}$

then $\|\vec{c}(t)\|^2 = \vec{c}(t)\cdot\vec{c}(t) = \text{const}$.

$$\Rightarrow (\vec{c}(t)\cdot\vec{c}(t))' = 0$$

$$\Rightarrow \vec{c}'(t)\cdot\vec{c}(t) + \vec{c}(t)\cdot\vec{c}'(t) = 0$$

$$2\,\vec{c}(t)\cdot\vec{c}'(t) = 0$$

so $\vec{c}'(t)$ is orth. to $\vec{c}(t)$ for all t.

Definition: $\vec{a}(t) = \vec{v}'(t) = \vec{c}''(t)$ is the acceleration of the curve.

$$\Rightarrow \vec{a}(t) = (x''(t), y''(t), z''(t))$$

Example: Suppose that the acceleration of a particle is $\vec{a}(t) = -\vec{k}\ \forall t$ (constant acc.)
Suppose that $\vec{c}(0) = (0,0,1)$ & $\vec{v}(0) = (1,1,0)$
Find the time a spatial coordinates when the particle reaches $z = 0$.

Soln: want to find t & $\vec{c}(t) = (x(t), y(t), z(t))$

where $z(t) = 0$.

we know that $\vec{a}(t) = (0, 0, -1) \quad \forall t$

& $\vec{a}(t) = (x''(t), y''(t), z''(t))$

but $\vec{v}(t) = (x'(t), y'(t), z'(t))$

So $x'(t) = \text{const}$

$y'(t) = \text{const}$

$z'(t) = -t + \text{const}$

But $\vec{x}'(0) = 1 \Rightarrow x'(t) = 1$

$y'(0) = 1 \Rightarrow y'(t) = 1$

$z'(0) = 0 \Rightarrow z'(t) = -t$

$\Downarrow$

Integrating again

$x(t) = t + \text{const}$ and $x(0) = 0 \Rightarrow x(t) = t$

$y(t) = t + \text{const}$ and $y(0) = 0 \Rightarrow y(t) = t$

$z(t) = -t^2/2 + \text{const}$ & $z(0) = 1 \Rightarrow z(t) = 1 - t^2/2$

We want t: $z(t) = 0$ so we solve

$1 - t^2/2 = 0 \Rightarrow t = \sqrt{2}$ (bec $t \geq 0$)

So $\vec{c}(\sqrt{2}) = (\sqrt{2}, \sqrt{2}, 0)$ is the position of the particle when it crosses the $z=0$ plane

Physics example (How long is a planet's year, knowing only its radius)

Suppose $\vec{c}(t) = \left(r\cos\frac{st}{r}, r\sin\frac{st}{r}\right)$ ← circular orbit

is the orbit of a planet.

Then, the period is $T = \frac{2\pi r}{s}$ ← time to complete revolution & $\|\vec{c}(t)\| = r$ ← radius of orbit

The planet's velocity is $\vec{v}(t) = \left(-s\sin\frac{st}{r}, s\cos\frac{st}{r}\right)$

and its speed is $\|\vec{v}(t)\| = \sqrt{(\ )^2 + (\ )^2} = s$

The acceleration is $\vec{a}(t) = \vec{v}'(t) = \left(-\frac{s^2}{r}\cos\frac{st}{r}, -\frac{s^2}{r}\sin\frac{st}{r}\right)$

So $\vec{a}(t) = -\frac{s^2}{r^2}\left(r\cos\frac{st}{r}, r\sin\frac{st}{r}\right)$

$-\omega^2$ (call $s^2/r^2 = \omega^2$) $\vec{c}(t)$ (ω is the frequency)

So $\vec{a}(t) = -\omega^2\,\vec{c}(t)$

But Physics tells us that force $\vec{F} = m\vec{a}$ (m: mass)

and that $\vec{F} = -\frac{GmM}{r^3}\vec{c}$ (G: const., M: mass of sun, m: mass of planet)

newton's law of gravity

So now $-m\omega^2\vec{c}(t) = -\frac{GmM}{r^3}\vec{c}(t)$

taking norms on both sides

$-m\frac{s^2}{r}\cancel{r} = -\frac{GmM}{r^{\cancel{3}2}}\cancel{r}$

so $s^2 = \frac{GM}{r}$

$s = \frac{2\pi r}{T}$

$\Rightarrow \boxed{T^2 = r^3\frac{(2\pi)^2}{GM}}$

Kepler's law